THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4060 Complex Analysis
Homework 4 Suggested Solutions
Date: 27 March, 2025

1. (Exercise 5 of Chapter 8 of [SS03]) Prove that $f(z) = -\frac{1}{2}(z + 1/z)$ is a conformal map from the half-disc $\{z = x + iy : |z| < 1, y > 0\}$ to the upper half-plane.

   [Hint: The equation $f(z) = w$ reduces to the quadratic equation $z^2 + 2wz + 1 = 0$, which has two distinct roots in $\mathbb{C}$ whenever $w \neq \pm 1$. This is certainly the case if $w \in \mathbb{H}$.]

   Solution. Since $y > 0$, $z \neq 0$ and we clearly see that f is holomorphic. We check that the image of f indeed lies in $\mathbb{H}$. Let $z = x + iy$ with $|z| < 1$ and $y > 0$, then

$$\begin{aligned}\operatorname{Im}(f(z)) &= \operatorname{Im}\left(-\frac{1}{2}\left(z + \frac{1}{z}\right)\right) = \operatorname{Im}\left(-\frac{1}{2}\left(x + iy + \frac{1}{x + iy}\right)\right) \\ &= \operatorname{Im}\left(-\frac{1}{2}\left(x + iy + \frac{x - iy}{|z|^2}\right)\right) = \frac{y}{2|z|^2} - \frac{y}{2} \\ &= \frac{y}{2}\left(\frac{1}{|z|^2} - 1\right) > 0\end{aligned}$$

   where the last inequality holds because $|z|^2 < 1$ and $y > 0$.

   We proceed to show that f is bijective. To show that f is injective, suppose $\operatorname{Im}(u) > 0$ and $\operatorname{Im}(v) > 0$ and $f(u) = f(v)$. Then we have

$$u + \frac{1}{u} = v + \frac{1}{v} \Rightarrow u^2v - v^2u + v - u = 0 \Rightarrow (v - u)(1 - uv) = 0$$

   so either $v = u$ or $1 = uv \Rightarrow v = \dfrac{1}{u}$. In the case where $v = \dfrac{1}{u}$, we write $u = x + iy$ where $y > 0$, then we have that $v = \dfrac{1}{x + iy} = \dfrac{x - iy}{x^2 + y^2}$ but then this means that $\operatorname{Im}(v) < 0$, which is a contradiction. Hence, only $u = v$ is possible and so f is injective.

   To show surjectivity, suppose $w \in \mathbb{H}$. We attempt to solve for $f(z) = w$, that is

$$-\frac{1}{2}\left(z + \frac{1}{z}\right) = w \Rightarrow z^2 + 2wz + 1 = 0.$$

   As in the hint, when $w \in \mathbb{H}$, the quadratic equation has two roots in z, say α, β. By Viéta's formulas, we have that $\alpha\beta = 1$ and $\alpha + \beta = -2w$. It suffices to show that $\alpha \in \mathbb{H}$. By $\alpha\beta = 1$, we have that $\beta = \dfrac{1}{\alpha}$, and we also see that either both $\alpha, 1/\alpha$ lie on the unit circle, or exactly one of $\alpha, 1/\alpha$ lies on the interior of the unit disk and the other lies on the exterior. Suppose both $\alpha, 1/\alpha$ lie on the unit disk, then we can easily see that $1/\alpha = \overline{\alpha}$ and then the equation $\alpha + \beta = \alpha + \overline{\alpha} = 2\operatorname{Re}(\alpha)$ contradicts the fact that $-2w$ has non-zero imaginary component. So we must have that one of $\alpha, 1/\alpha$ lie in the interior of the unit disk, say, α. Since we have $|\alpha| < 1$, we have that

$$\begin{aligned}0 > -2\mathrm{Im}(w) &= \mathrm{Im}(\alpha+\beta) = \mathrm{Im}\left(\alpha + \frac{1}{\alpha}\right)\\ &= \mathrm{Im}\left(\alpha + \frac{1}{\alpha}\frac{\overline{\alpha}}{\overline{\alpha}}\right) = \mathrm{Im}\left(\alpha + \frac{\overline{\alpha}}{|\alpha|^2}\right) = \mathrm{Im}(\alpha)\left(1 - \frac{1}{|\alpha|^2}\right)\end{aligned}$$

forces $\mathrm{Im}(\alpha) > 0$ since $1 - 1/|\alpha|^2 < 0$. Hence, $f(\alpha) = w$ with $\alpha \in \mathbb{H}$ and f is indeed surjective. ◀

2. (Exercise 6 of Chapter 8 of [SS03]) Give another proof of Lemma 1.3 by showing directly that the Laplacian of $u \circ F$ is zero.

   [Hint: The real and imaginary parts of F satisfy the Cauchy-Riemann equations.]

   Solution. We prove the statement: Let V and U be open sets in $\mathbb{C}$ and $F : V \to U$ a holomorphic function. If $u : U \to \mathbb{C}$ is a harmonic function, then $u \circ F$ is harmonic on V.

   We first show that for $z = x + iy$ and $F(z) = F(x, y) = a(x, y) + ib(x, y)$ is holomorphic in V, then both a and b are harmonic. Note that since F is holomorphic, it is infinitely differentiable and hence a, b are also infinitely differentiable, in particular, possessing continuous second partial-derivatives with identical mixed second-order partial derivatives. By the Cauchy-Riemann equations:

$$a_x = b_y, \quad a_y = -b_x$$

   and hence

$$\begin{aligned}a_{xx} + a_{yy} &= (a_x)_x + (a_y)_y = (b_y)_x + (-b_x)_y = b_{yx} - b_{xy} = 0\\ b_{xx} + b_{yy} &= (b_x)_x + (b_y)_y = (-a_y)_x + (a_x)_y = -a_{yx} + a_{xy} = 0.\end{aligned}$$

   Moreover by the Cauchy-Riemann equations, we also have

$$a_x b_x = -a_y b_y$$

   Now suppose $u(a, b)$ is harmonic on U. Then we compute $\Delta(u \circ F)$. We have

$$u_x = u_a a_x + u_b b_x, \quad u_y = u_a a_y + u_b b_y$$

   and

$$\begin{aligned}u_{xx} + u_{yy} &= u_{aa}(a_x)^2 + u_{ab}a_x b_x + u_a a_{xx} + u_{bb}(b_x)^2 + u_{ba}b_x a_x + u_b b_{xx}\\ &\quad + u_{aa}(a_y)^2 + u_{ab}a_y b_y + u_a a_{yy} + u_{bb}(b_y)^2 + u_{ba}b_y a_y + u_b b_{yy}\\ &= u_{aa}(a_x)^2 - u_{ab}a_y b_y + u_a(a_{xx} + a_{yy}) + u_{bb}(b_x)^2 - u_{ba}b_y a_y + u_b(b_{xx} + b_{yy})\\ &\quad + u_{aa}(a_y)^2 + u_{ab}a_y b_y + u_{bb}(b_y)^2 + u_{ba}b_y a_y\\ &= u_{aa}(a_x)^2 + u_{bb}(b_x)^2 + u_{aa}(a_y)^2 + u_{bb}(b_y)^2\\ &= u_{aa}(b_y)^2 + u_{bb}(b_x)^2 + u_{aa}(-b_x)^2 + u_{bb}(b_y)^2\\ &= (b_x)^2(u_{aa} + u_{bb}) + (b_y)^2(u_{aa} + u_{bb}) = 0\end{aligned}$$

   where in the last equality we are using the fact that u is harmonic. Hence $u \circ F$ is harmonic in V. ◀

3. (Exercise 11 of Chapter 8 of [SS03]) Show that if $f : D(0,R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \leq M$ for some $M > 0$, then
$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)} \right| \leq \frac{|z|}{MR}.$$

Solution. If f is constant, the inequality is trivial. Hence, from now on we suppose f is not constant. Then by the maximum modulus principle, f does not achieve M anywhere on $D(0,R)$, hence, the image of f lies in $D(0,M)$. Therefore, the function g defined by
$$g(w) = \frac{\frac{f(Rw)}{M} - \frac{f(0)}{M}}{1 - \frac{\overline{f(0)}}{M}\frac{f(Rw)}{M}} = M\frac{f(Rw) - f(0)}{M^2 - \overline{f(0)}f(Rw)}$$
is a holomorphic function from $\mathbb{D}$ to $\mathbb{D}$. Moreover, we can see that $g(0) = 0$, and so by the Schwarz Lemma, we have that $|g(w)| \leq |w|$. This gives
$$\left| M\frac{f(Rw) - f(0)}{M^2 - \overline{f(0)}f(Rw)} \right| \leq |w| \Rightarrow \left| \frac{f(Rw) - f(0)}{M^2 - \overline{f(0)}f(Rw)} \right| \leq \frac{|w|}{M}$$
and finally putting $z = Rw$ yields the desired inequality. ◀

4. (Exercise 14 of Chapter 8 of [SS03]) Prove that all conformal mappings from the upper half-plane $\mathbb{H}$ to the unit disk $\mathbb{D}$ take the form
$$e^{i\theta}\frac{z - \beta}{z - \overline{\beta}}, \qquad \theta \in \mathbb{R} \text{ and } \beta \in \mathbb{H}.$$

Solution. Let $f : \mathbb{H} \to \mathbb{D}$ be a conformal mapping. Recall that $F(z) = \dfrac{i - z}{i + z}$ is a conformal map from $\mathbb{H}$ to $\mathbb{D}$. Then we see that $f \circ F^{-1} : \mathbb{D} \to \mathbb{D}$ is an automorphism and hence by Theorem 2.2 of Chapter 8 of [SS03], takes the form
$$(f \circ F^{-1})(z) = e^{i\theta}\frac{\alpha - z}{1 - \overline{\alpha}z}$$
for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. Then we see that
$$\begin{aligned} f(z) = (f \circ F^{-1} \circ F)(z) &= (f \circ F^{-1})\left(\frac{i - z}{i + z}\right) \\ &= e^{i\theta}\frac{\alpha - \frac{i-z}{i+z}}{1 - \overline{\alpha}\frac{i-z}{i+z}} \\ &= e^{i\theta}\frac{z(\alpha + 1) - i(1 - \alpha)}{z(\overline{\alpha} + 1) + i(1 - \overline{\alpha})} \\ &= e^{i\theta}\frac{z - \beta}{z - \overline{\beta}}\frac{\alpha + 1}{\overline{\alpha} + 1} \\ &= e^{i(\theta + \varphi)}\frac{z - \beta}{z - \overline{\beta}} \end{aligned}$$
where $\beta = \dfrac{i(1 - \alpha)}{\alpha + 1}$ and one can verify that $\overline{\beta} = \dfrac{-i(1 - \overline{\alpha})}{\overline{\alpha} + 1}$ and we are using the fact that $\dfrac{\alpha + 1}{\overline{\alpha} + 1}$ has norm one and hence can be represented as a rotation $e^{i\varphi}$. ◀

References

[SS03] Elias M. Stein and Rami Shakarchi. *Complex Analysis*. Princeton University Press, 2003.